Robust Multi-Image HDR Reconstruction for the Modulo Camera

– Supplemental Material –

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Preface. This supplementary material provides additional mathematical derivations. Specifically, we derive the explicit expression of the image differences e_i (Eqs. 18 and 19), we prove Theorem 1 on the correctness of our robust algorithm, and prove that the original reconstruction algorithm from [22] is correct only for the case of no noise and carefully set exposure times. Moreover, we give a derivation of the exposure time schedule (Eq. 24) as well as the theoretical upper bound τ^* on the maximal achievable exposure time in the presence of image noise.

Notation. We quickly recap some of the notation used in the main paper. The image taken with a hypothetical, conventional K-bit camera and exposure time τ_i is defined as

$$I_i = \lfloor \tau_i R + \epsilon(\tau_i R) \rfloor = k_i 2^L + M_i. \tag{28}$$

The reconstructed image from our robust algorithm at time step i is given by

$$\tilde{I}_i = \lfloor \tau_i \tilde{R}_i \rfloor = \tau_i \tilde{R}_i = (\tilde{k}_i + \Delta_i) 2^L + M_i. \tag{29}$$

Note that the lower L bits of Eqs. (28) and (29) agree, because a L-bit modulo camera captures the L least significant bits. For the following analysis it is convenient to define a simulated reconstruction $\bar{I}_i$ that uses the approximated radiance map $\tilde{R}_{i-1}$ from the previous time step instead of the updated radiance map $\tilde{R}_i$

$$\bar{I}_i = \lfloor \tau_i \tilde{R}_{i-1} \rfloor \doteq \tilde{k}_i 2^L + D_i. \tag{30}$$

Please note that D_i is a proper modulo image, *i.e.* $0 \leq D_i < 2^L$. This follows from the definition of $\tilde{k}_i$ (Eq. 6):

$$\tilde{k}_i = \left\lfloor \frac{\tau_i \tilde{R}_{i-1}}{2^L} \right\rfloor. \tag{31}$$

A Derivation of e_i (Eq. 19)

We now derive the explicit form of e_i given by Eq. (19):

$$
\begin{aligned}
e_i =& I_i - \frac{\tau_i}{\tau_{i-1}} I_{i-1} && (32)\\
=& \lfloor \tau_i R + \epsilon(\tau_i R) \rfloor - \frac{\tau_i}{\tau_{i-1}} \lfloor \tau_{i-1} R + \epsilon(\tau_{i-1} R) \rfloor && (33)\\
=& \tau_i R + \epsilon(\tau_i R) - r\big(\tau_i R + \epsilon(\tau_i R)\big) && (34)\\
& - \frac{\tau_i}{\tau_{i-1}} \Big(\tau_{i-1} R + \epsilon(\tau_{i-1} R) - r\big(\tau_{i-1} R + \epsilon(\tau_{i-1} R)\big)\Big) &&\\
=& \epsilon(\tau_i R) - \frac{\tau_i}{\tau_{i-1}} \epsilon(\tau_{i-1} R) - \Big(\underbrace{r\big(\tau_i R + \epsilon(\tau_i R)\big)}_{\doteq r_i} - \frac{\tau_i}{\tau_{i-1}} \underbrace{r\big(\tau_{i-1} R + \epsilon(\tau_{i-1} R)\big)}_{\doteq r_{i-1}}\Big) && (35)\\
=& \epsilon(\tau_i R) - \frac{\tau_i}{\tau_{i-1}} \epsilon(\tau_{i-1} R) - \Big(r_i - \frac{\tau_i}{\tau_{i-1}} r_{i-1}\Big). && (36)
\end{aligned}
$$

Here, we model the effect of rounding as explicit rounding errors r_i, r_{i-1} with $r(x) = x - \lfloor x \rfloor$ and $0 \leq r(x) < 1$. This allows us to work without floor functions in the following. As we can see, e_i is given by the difference of both image noise terms as well as the difference of both rounding error terms.

B Proof of Theorem 1

We now prove the correctness of our robust reconstruction algorithm under the assumption that $|e_i| \leq 2^{L-1} - 1$. We use induction to show that $\tilde{I}_i = I_i$ for all $i = 1, \ldots, n$. The induction basis holds, since $\tilde{I}_1 = M_1 = I_1$ due to the initialization of our algorithm and the assumption that the exposure time τ_1 is short enough such that the first image has no rollovers. For $i > 1$ we note that

$$
\begin{aligned}
& \tilde{I}_i = I_i && (37)\\
\Leftrightarrow \quad & (\tilde{k}_i + \Delta_i) 2^L + M_i = k_i 2^L + M_i && (38)\\
\Leftrightarrow \quad & (\tilde{k}_i + \Delta_i) = k_i. && (39)
\end{aligned}
$$

The least significant bits are equal by construction and it suffices to show that the leading bits, defined by the number of rollovers, are equal. To do that, we will proceed in two steps. First, we show that e_i can be expressed in terms of k_i and $\tilde{k}_i$ as

$$
e_i = (k_i - \tilde{k}_i) 2^L + (M_i - D_i) - r', \tag{40}
$$

where r' is a rounding error term. Then, we show that assuming $(\tilde{k}_i + \Delta_i) \neq k_i$ implies that $|e_i| > 2^{L-1} - 1$, which contradicts our assumption that $|e_i| \leq 2^{L-1} - 1$. Hence, Eq. (39) must hold, which in turn implies Eq. (37).

Step 1

We derive e_i in another way by looking at the difference between $\bar{I}_i$ and I_i, which we define as $\bar{e}_i$:

$$\bar{e}_i \doteq I_i - \bar{I}_i \tag{41}$$
$$= (k_i - \tilde{k}_i)2^L + (M_i - D_i) \tag{42}$$

where we used the definitions of I (Eq. 28) and $\bar{I}$ (Eq. 30). We now show that $\bar{e}_i$ and e_i are equal up to a rounding error, *i.e.* $\bar{e}_i = e_i + r'$, which concludes the first step (Eq. 40). By using the definitions of I_i and $\bar{I}_i$ we have

$$\bar{e}_i = I_i - \bar{I}_i = \lfloor \tau_i R + \epsilon(\tau_i R) \rfloor - \lfloor \tau_i \tilde{R}_{i-1} \rfloor \tag{43}$$
$$= \tau_i R + \epsilon(\tau_i R) - r_i - \left\lfloor \frac{\tau_i}{\tau_{i-1}} \tau_{i-1} \tilde{R}_{i-1} \right\rfloor. \tag{44}$$

Now, we replace $\tilde{I}_{i-1} = \tau_{i-1}\tilde{R}_{i-1}$ with $I_{i-1} = \lfloor \tau_{i-1} R + \epsilon(\tau_{i-1} R) \rfloor$ by using the induction assumption

$$\bar{e}_i = \tau_i R + \epsilon(\tau_i R) - r_i - \left\lfloor \frac{\tau_i}{\tau_{i-1}} \lfloor \tau_{i-1} R + \epsilon(\tau_{i-1} R) \rfloor \right\rfloor \tag{45}$$
$$= \tau_i R + \epsilon(\tau_i R) - r_i - \left\lfloor \tau_i R + \frac{\tau_i}{\tau_{i-1}} \epsilon(\tau_{i-1} R) - \frac{\tau_i}{\tau_{i-1}} r_{i-1} \right\rfloor \tag{46}$$
$$= \tau_i R + \epsilon(\tau_i R) - r_i - \tau_i R - \frac{\tau_i}{\tau_{i-1}} \epsilon(\tau_{i-1} R) + \frac{\tau_i}{\tau_{i-1}} r_{i-1} \tag{47}$$
$$+ \underbrace{r(\tau_i R + \frac{\tau_i}{\tau_{i-1}} \epsilon(\tau_{i-1} R) - \frac{\tau_i}{\tau_{i-1}} r_{i-1})}_{=:r'} \tag{48}$$
$$= e_i + r'. \tag{49}$$

Step 2

We want to show the equality of rollovers (Eq. 39) by contradiction. Since $k_i, \tilde{k}_i, \varDelta_i \in \mathbb{Z}$ it holds

$$(\tilde{k}_i + \varDelta_i) \neq k_i \tag{50}$$
$$\Leftrightarrow \Big(k_i - \tilde{k}_i \geq \varDelta_i + 1 \quad \text{or} \quad k_i - \tilde{k}_i \leq \varDelta_i - 1 \Big). \tag{51}$$

We will show that

$$k_i - \tilde{k}_i \geq \varDelta_i + 1 \implies e_i > 2^{L-1} - 1 \tag{52}$$

and

$$k_i - \tilde{k}_i \leq \varDelta_i - 1 \implies e_i < -(2^{L-1} - 1) \tag{53}$$

leading to a contradiction since we assumed that $|e_i| \leq 2^{L-1} - 1$. To show Eqs. (52) and (53) we make a case distinction on $\varDelta_i \in \{-1, 0, 1\}$.

Case 1: $\boldsymbol{\Delta_i = 0}$. For $\Delta_i = 0$ we know from Eq. (17) that $-2^{L-1} \leq (M_i - D_i) \leq 2^{L-1}$. Now, for $k_i - \tilde{k}_i \geq \Delta_i + 1 = 1$ we have

$$e_i = (k_i - \tilde{k}_i)2^L + (M_i - D_i) - r' \tag{54}$$
$$> 2^L - 2^{L-1} - 1 \tag{55}$$
$$= 2^{L-1} - 1. \tag{56}$$

Analogously, for $k_i - \tilde{k}_i \leq \Delta_i - 1 = -1$ we have

$$e_i = (k_i - \tilde{k}_i)2^L + (M_i - D_i) - r' \tag{57}$$
$$\leq -2^L + 2^{L-1} - 0 \tag{58}$$
$$= -2^{L-1}. \tag{59}$$

Case 2: $\boldsymbol{\Delta_i = 1}$. For $\Delta_i = 1$ we know that $-2^L + 1 \leq (M_i - D_i) \leq -2^{L-1} - 1$, with the left inequality following from M_i and D_i being modulo values and the right inequality following from Eq. (17). Now, for $k_i - \tilde{k}_i \geq \Delta_i + 1 = 2$ we have

$$e_i = (k_i - \tilde{k}_i)2^L + (M_i - D_i) - r' \tag{60}$$
$$> 2 \cdot 2^L - 2^L + 1 - 1 \tag{61}$$
$$= 2^L. \tag{62}$$

For the other case $k_i - \tilde{k}_i \leq \Delta_i - 1 = 0$ we have

$$e_i = (k_i - \tilde{k}_i)2^L + (M_i - D_i) - r' \tag{63}$$
$$\leq 0 - 2^{L-1} - 1 - 0 \tag{64}$$
$$= -2^{L-1} - 1. \tag{65}$$

Case 3: $\boldsymbol{\Delta_i = -1}$. Similar to the last case, we can derive the two conditions

$$e_i > 2^{L-1} \quad \text{for} \quad k_i - \tilde{k}_i \geq \Delta_i + 1 = 0 \tag{66}$$
$$e_i \leq -2^L - 1 \quad \text{for} \quad k_i - \tilde{k}_i \leq \Delta_i - 1 = -2. \tag{67}$$

Combination of cases. Since we do not know a priori which of these three cases will occur, we need to pick the loosest condition. For $k_i - \tilde{k}_i \geq \Delta_i + 1$ we have

$$e_i > \min(2^{L-1} - 1, 2^L, 2^{L-1}) = 2^{L-1} - 1. \tag{68}$$

Similarly, for $k_i - \tilde{k}_i \leq \Delta_i - 1$ we have

$$e_i \leq \max(-2^{L-1}, -2^{L-1} - 1, -2^L - 1) = -2^{L-1}. \tag{69}$$

Hence, we see that in both cases even the loosest condition on e_i violates our assumption that $|e_i| \leq 2^{L-1} - 1$. $\square$

C Correctness of the Original Algorithm

Let us first present a simple yet not so obvious inequality concerning rounding errors.

Lemma 1. *For any $x \in \mathbb{R}$ and $c \in \mathbb{N}$ we have $r\left(\frac{x}{c}\right) \geq \frac{1}{c} r(x)$.*

Proof. We first apply simple transformations to the inequality:

$$r\left(\frac{x}{c}\right) \geq \frac{1}{c} r(x) \tag{70}$$

$$\Leftrightarrow \quad \frac{x}{c} - \left\lfloor \frac{x}{c} \right\rfloor \geq \frac{1}{c}(x - \lfloor x \rfloor) \tag{71}$$

$$\Leftrightarrow \quad \lfloor x \rfloor \geq c \left\lfloor \frac{x}{c} \right\rfloor . \tag{72}$$

The last inequality is true as for any $x, y \in \mathbb{R}$ it holds that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor$. Since $c \in \mathbb{N}$ we can apply this inequality c times to obtain

$$c \left\lfloor \frac{x}{c} \right\rfloor \leq \left\lfloor c \frac{x}{c} \right\rfloor = \lfloor x \rfloor . \tag{73}$$

□

Now, we can prove the correctness of the original multishot reconstruction algorithm given noise-free observations and certain constraints on the exposure time schedule.

Theorem 2. *The reconstruction of the original algorithm of [22] is provably exact,* i.e. $\tilde{I}_i = I_i$, *if the fraction $2^L \frac{\tau_{i-1}}{\tau_i}$ is a positive integer,* i.e. *there exists $c \in \mathbb{N}$ with $c = 2^L \frac{\tau_{i-1}}{\tau_i}$*, and *if the recorded images are not corrupted by noise,* i.e. $I_i = \lfloor \tau_i R \rfloor$, *for all $i = 1, \dots, n$.*

Proof. We proceed by induction. For $i = 1$ we have $\tilde{I}_1 = I_1$ by construction when assuming that the first image has no rollovers. For $i > 1$ we again show that $\tilde{k}_i = k_i$, which implies that $\tilde{I}_i = I_i$, *c.f.* Appendix B. We can now rewrite $\tilde{k}_i$ as

$$\tilde{k}_i = \left\lfloor \frac{\tau_i \tilde{R}_{i-1}}{2^L} \right\rfloor = \left\lfloor \frac{\tau_i}{\tau_{i-1}} \frac{\tau_{i-1} \tilde{R}_{i-1}}{2^L} \right\rfloor = \left\lfloor \frac{\tau_i}{2^L \tau_{i-1}} \lfloor \tau_{i-1} R \rfloor \right\rfloor , \tag{74}$$

where we used the induction hypothesis in the last step. Let us now further rewrite Eq. (74) by introducing explicit terms for the rounding error.

$$\tilde{k}_i = \left\lfloor \frac{\tau_i R}{2^L} - \frac{1}{c} r(\tau_{i-1} R) \right\rfloor = \left\lfloor \left\lfloor \frac{\tau_i R}{2^L} \right\rfloor + r\left(\frac{\tau_i R}{2^L} \right) - \frac{1}{c} r(\tau_{i-1} R) \right\rfloor \tag{75}$$

$$= \left\lfloor \left\lfloor \frac{\tau_i R}{2^L} \right\rfloor + r\left(\frac{\tau_{i-1} R}{c} \right) - \frac{1}{c} r(\tau_{i-1} R) \right\rfloor . \tag{76}$$

From the first equality it becomes clear that $\tilde{k}_i \leq k_i$, since

$$\tilde{k}_i = \left\lfloor \frac{\tau_i R}{2^L} - \frac{1}{c} r(\tau_{i-1} R) \right\rfloor \leq \left\lfloor \frac{\tau_i R}{2^L} \right\rfloor = k_i. \tag{77}$$

For the other direction, *i.e.* $\tilde{k}_i \geq k_i$, it suffices to show that

$$r\left(\frac{\tau_{i-1} R}{c}\right) \geq \frac{1}{c} r(\tau_{i-1} R). \tag{78}$$

This is true by invoking Lemma 1. Plugging the last inequality into Eq. (76) we see that

$$\tilde{k}_i = \left\lfloor \left\lfloor \frac{\tau_i R}{2^L} \right\rfloor + r\Big(\frac{\tau_{i-1} R}{c}\Big) - \frac{1}{c} r(\tau_{i-1} R) \right\rfloor \geq \left\lfloor \left\lfloor \frac{\tau_i R}{2^L} \right\rfloor \right\rfloor = k_i. \tag{79}$$

From Eqs. (77) and (79) we conclude that $\tilde{k}_i = k_i$. □

Failure case. We now give an example (assuming $L = 8$) to show that the original algorithm might fail even when the assumption on the τ_i is violated only slightly. Let $R = 256$, $\tau_1 = 0.4$ and $\tau_2 = 1$. Then, the true images are $I_1 = \lfloor 0.4R \rfloor = 102$ and $I_2 = \lfloor R \rfloor = 256$. The modulo images are $M_1 = 102$ and $M_2 = 0$. Executing the original algorithm yields

$$\tilde{R}_1 = \frac{M_1}{\tau_1} = 255 \tag{80}$$

$$\tilde{k}_2 = \left\lfloor \frac{255}{256} \right\rfloor = 0 \tag{81}$$

$$\tilde{R}_2 = \tilde{k}_2 2^8 + M_2 = 0 + 0 = 0. \tag{82}$$

Therefore, the final reconstruction $\tilde{I}_2 = 0 \neq 256 = I_2$ deviates from the true image by a wide margin.

D Derivation of Robust Exposure Time Schedule (Eq. 24)

We first simplify the distribution of image differences e_i. By inserting the intensity-dependent noise distributions Eq. (2) for ϵ in Eq. (36), we get

$$e_i \sim \mathcal{N}(0, \beta_1 \tau_i R + \beta_2) - \frac{\tau_i}{\tau_{i-1}} \mathcal{N}(0, \beta_1 \tau_{i-1} R + \beta_2) - r_i + \frac{\tau_i}{\tau_{i-1}} r_{i-1} \tag{83}$$

$$\sim \mathcal{N}\Bigg(0, \underbrace{\beta_1 \tau_i R\big(1 + \tfrac{\tau_i}{\tau_{i-1}}\big) + \beta_2\big(1 + \tfrac{\tau_i^2}{\tau_{i-1}^2}\big)}_{=\sigma_i^2}\Bigg) - r_i + \frac{\tau_i}{\tau_{i-1}} r_{i-1}. \tag{84}$$

According to Eq. (21) of the main paper, we want to find the maximal τ_i such that the reconstruction is correct with probability of at least p, *i.e.*

$$\mathrm{P}\big[|e_i| \leq 2^{L-1} - 1\big] \geq p. \tag{85}$$

We can now further bound the left-hand side of above equation by using the triangle inequality and exploiting that $0 \leq r_i < 1$ and $0 \leq r_{i-1} < 1$:

$$\mathrm{P}\left[|e_i| \leq 2^{L-1} - 1\right] = \mathrm{P}\left[\left|\mathcal{N}(0, \sigma_i^2) - r_i + \frac{\tau_i}{\tau_{i-1}} r_{i-1}\right| \leq 2^{L-1} - 1\right] \tag{86}$$

$$\geq \mathrm{P}\left[\left|\mathcal{N}(0, \sigma_i^2)\right| + \left|\frac{\tau_i}{\tau_{i-1}} r_{i-1} - r_i\right| \leq 2^{L-1} - 1\right] \tag{87}$$

$$\geq \mathrm{P}\left[\left|\mathcal{N}(0, \sigma_i^2)\right| \leq 2^{L-1} - 1 - \frac{\tau_i}{\tau_{i-1}}\right]. \tag{88}$$

In the last line we used the fact that ${}^{\tau_i}\!/\!{}_{\tau_{i-1}} > 1$ to bound r_i and r_{i-1} with 0 and 1, respectively. We now use the last expression to derive a slightly stricter criterion on the τ_i than Eq. (21):

$$\mathrm{P}\left[\left|\mathcal{N}(0, \sigma_i^2)\right| \leq 2^{L-1} - 1 - \frac{\tau_i}{\tau_{i-1}}\right] \geq p \tag{89}$$

$$\Leftrightarrow \quad \mathrm{P}\left[\mathcal{N}(0, \sigma_i^2) \leq 2^{L-1} - 1 - \frac{\tau_i}{\tau_{i-1}}\right] \geq \tfrac{1}{2} + \tfrac{1}{2}p \tag{90}$$

$$\Leftrightarrow \quad 2^{L-1} - 1 - \frac{\tau_i}{\tau_{i-1}} \geq \sigma_i \Phi^{-1}\left(\tfrac{1}{2} + \tfrac{1}{2}p\right), \tag{91}$$

where we used the symmetry of $\mathcal{N}(0, \sigma_i^2)$ and the fact that the inverse CDF of $\mathcal{N}(0, \sigma^2)$ is monotonically increasing and given by $\sigma\Phi^{-1}(p)$ with $\Phi^{-1}(p)$ being the inverse CDF of $\mathcal{N}(0, 1)$. By denoting $l = {}^{1}\!/\!{}_{\Phi^{-1}\left(\frac{1}{2}+\frac{1}{2}p\right)}$ we obtain the following condition on σ_i^2 such that the inequality (Eq. 91) is exact:

$$\sigma_i^2 = l^2(2^{L-1} - 1)^2 + l^2\left(\frac{\tau_i}{\tau_{i-1}}\right)^2 - 2l^2\frac{\tau_i}{\tau_{i-1}}(2^{L-1} - 1). \tag{92}$$

After plugging in the definition of σ_i^2 (Eq. 84) we can reorder terms in numerous simple steps that we omit for brevity. We finally arrive at a quadratic equation for the ratio of τ_i and τ_{i-1}

$$a\left(\frac{\tau_i}{\tau_{i-1}}\right)^2 + b\frac{\tau_i}{\tau_{i-1}} + c = 0 \tag{93}$$

with

$$a = \beta_1 \tau_{i-1} R + \beta_2 - l^2 \tag{94}$$

$$b = \beta_1 \tau_{i-1} R + l^2(2^L - 2) \tag{95}$$

$$c = \beta_2 - l^2(2^{L-1} - 1)^2. \tag{96}$$

Solving this quadratic equation yields

$$\left(\frac{\tau_i}{\tau_{i-1}}\right)_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{97}$$

Keeping only the positive exposure time yields the final recursion formula

$$\tau_i \leq \tau_{i-1} \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{98}$$

E Explicit Formula for τ^*

To obtain a theoretical upper bound on the exposure time τ^* we set $\tau_i = \tau_{i-1} = \tau^*$, *i.e.* τ^* is the fixed point of the recursion formula (Eq. 98). This yields

$$\tau^* = \tau^* \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{99}$$

$$\Leftrightarrow \quad 1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{100}$$

$$\Leftrightarrow \quad a(a + b + c) = 0 \tag{101}$$

$$\Leftrightarrow \quad \big((\beta_1 \tau^* R) + \beta_2 - l^2\big)\big[\big((\beta_1 \tau^* R) + \beta_2 - l^2\big) \tag{102}$$

$$+ \big((\beta_1 \tau^* R) + l^2(2^L - 2)\big) + \big(\beta_2 - l^2(2^{L-1} - 1)^2\big)\big] = 0, \tag{103}$$

where we inserted Eqs. (94) to (96) at the last step. Again, a simple but tedious rearrangement of terms yields a quadratic equation in τ^*:

$$(\tau^*)^2 A + \tau^* B + C = 0, \tag{104}$$

with

$$A = 2\beta_1^2 R^2 \tag{105}$$

$$B = 4\beta_1\beta_2 R + (2^{L+1} - 6 - 2^{2L-2})\beta_1 R l^2 \tag{106}$$

$$C = 2\beta_2^2 + l^4(2^{2L-2} - 2^{L-1} + 4) + \beta_2 l^2(2^{L+1} - 2^{2L-2} - 2). \tag{107}$$

Now, we can solve Eq. (104) for τ^* yielding

$$\tau^*_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{108}$$

Again we are only interested in a positive τ^* and therefore

$$\tau^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{109}$$